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POMPEIU TRANSFORMS ON GEODESIC SPHERES IN REAL ANALYTIC MANIFOLDS

BY

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ABSTRACT

We prove a support theorem for Pompeiu transforms integrating on geodesic spheres of fixed radius $r > 0$ on real analytic manifolds when the measures are real analytic and nowhere zero. To avoid pathologies, we assume that r is less than the injectivity radius at the center of each sphere being integrated over. The proof of the main result is local and it involves the microlocal properties of the Pompeiu transform and a theorem of Hörmander, Kawai, and Kashiwara on microlocal singularities.

1. Introduction

The Pompeiu transform in $\mathbb{R}^n$ integrates functions over rigid motions of one or more fixed sets, and an intriguing special case is the transform that integrates over spheres. If $f \in L^1(\mathbb{R}^n)$ has zero integrals over translates of one sphere centered at the origin, S, ($f * m_S = 0$ where m_S is the standard measures on S), then a simple Fourier transform argument shows $f \equiv 0$. Our proof below is, at least morally, a microlocal version of this argument. In general, integrals over spheres of one radius do not determine f, but integrals over spheres of two "well chosen" radii determine $f \in C(\mathbb{R}^n)$ [7, 24, cf. 25]. This was generalized to symmetric spaces of real rank one in [4]. Properties including mean value theorems, injectivity, and inversion formulas were proven by Asgeirsson and John [12] in $\mathbb{R}^n$, and by Helgason [10] for homogeneous spaces. Local inversion formulas and support theorems are known for the Pompeiu transforms with standard weights

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on disks in symmetric spaces [19], on disks of two radii in $\mathbb{R}^n$ [1, 3], and for other sets in $\mathbb{R}^n$ [2]. The article [25] is a pleasing introduction to Pompeiu transforms and [26] provides a comprehensive bibliography and a current summary of results. These results are, in general, proven using the harmonic analysis of the specific ambient spaces. Using Fourier integral operator techniques that are local (actually microlocal), we prove support theorems for the Pompeiu transform that integrates over spheres in arbitrary real analytic manifolds.

Let M be a real analytic Riemannian manifold and let $d(x,y)$ be the Riemannian distance on M. Now, let $x \in M$ and $A \subset M$ be nonempty, then the distance from x to A is defined to be $d(x,A) = \inf\{d(x,y) \mid y \in A\}$. Let $r > 0$ be fixed. For $y \in M$ define $S(y)$ to be the geodesic sphere of radius r centered at y, $S(y) = \{x \in M \mid d(x,y) = r\}$, and let $D(y)$ denote the closed disk $D(y) = \{x \in M \mid d(x,y) \le r\}$. Let $y \in M$ and let I_y denote the injectivity radius of the exponential map at y.* If $I_y > r$, then $S(y)$ is the boundary of $D(y)$ and $S(y)$ is the diffeomorphic image under the exponential map of the Euclidean sphere of radius r centered at the origin in the tangent space T_yM [14, IV 3.4].

Let $\mathcal{A}$ be an open subset of M. Define $B(\mathcal{A},r) = \cup_{x\in\mathcal{A}} D(x)$ and note that, since $\mathcal{A}$ is open, so is $B(\mathcal{A},r)$. Assume that $r < I_y$ for each $y \in B(\mathcal{A},r)$. Let $\mu(x,y)$ be a nowhere zero real analytic function on

$$Z = \{(x,y) \in M \times \mathcal{A} \mid d(x,y) = r\}.$$

Let f be a continuous function on M, then the **Pompeiu transform** of f is defined for $y \in \mathcal{A}$ by

$$P_\mu f(y) = \int_{x\in S(y)} f(x)\mu(x,y)\,dm_S(x) \tag{1.1}$$

where dm_S is the canonical measure on $S(y)$ induced from the Riemannian structure on M. This is the integral of f over the geodesic sphere $S(y)$ with respect to the analytic measure μdm_S. Under these assumptions, $P_\mu^*\colon \mathcal{D}(\mathcal{A}) \to \mathcal{D}(M)$ is continuous. Therefore, $P_\mu\colon \mathcal{D}'(M) \to \mathcal{D}'(\mathcal{A})$ by duality.

* Let N be the largest connected open neighborhood of the zero section in TM such that the map $\varphi\colon N \to M \times M$, $\varphi(x,X) = (x, \exp_x X)$, is real analytic (see [14, III 8.1]). Let $B(x_0,s)$ be the open ball about $0 \in T_{x_0}M$ of radius $s > 0$. Then, for $x_0 \in M$, the **injectivity radius** I_{x_0} is the largest number s such that $B(x_0,s) \subset N$ and $\exp = \exp_{x_0}$ is a diffeomorphism on $B(x_0,s)$. From this definition, it is easy to see that φ is a diffeomorphism on a neighborhood in N of $B(x_0,I_{x_0})$ and so I_x is a lower semicontinuous function for $x \in M$.

We include the weight μ in the definition of the Pompeiu transform, (1.1), because canonical measures do not occur in general. Allowing non-standard measures can help focus on properties intrinsic to the Radon transform rather than on specific symmetry relations that are valid only for special cases. Also, our proofs, involving microlocal analysis, do not require canonical measures, and the canonical measure on $S(y)$ (and $S(y)$ itself) depends on the specific Riemannian structure on M. Our main theorem is:

THEOREM 1.1: *Let M be a real analytic Riemannian manifold and let $\mathcal{A} \subset M$ be open and connected. Let $r > 0$ and assume for each $y \in B(\mathcal{A}, r)$, $I_y > r$. Let P_μ be a Pompeiu transform on geodesic spheres in M of radius r with nowhere zero real analytic weight μ. Assume $f \in \mathcal{D}'(M)$ with $P_\mu f(y) = 0$ for all $y \in \mathcal{A}$ and assume, for some $y_0 \in \mathcal{A}$, the disk $D(y_0)$ is disjoint from* supp f. *Then for all $y \in \mathcal{A}$, $D(y)$ is disjoint from* supp f.

The assumption that $I_y > r$ for each $y \in B(\mathcal{A}, r)$ is a simple way to ensure that the Pompeiu transform (1.1) is defined on domain $\mathcal{D}'(M)$. This can be weakened to $I_y > r$ for each $y \in \mathcal{A}$ as long as $S(y) \cap \mathcal{A}$ is not pathological when $y \in B(\mathcal{A}, r)$. Theorem 1.1 does not contradict the non-injectivity of the Pompeiu transform on spheres of one radius in $\mathbb{R}^n$ or the non-injectivity of certain spherical cap transforms on S^{n-1} [17] because of the extra hypothesis in Theorem 1.1 that f must be zero near some $D(y_0)$. A counterexample, 3.2, is given if the hypothesis of Theorem 1.1, "*$D(y_0)$ is disjoint from supp f*," is weakened to become "*$S(y_0)$ is disjoint from supp f*."

Because non-compact symmetric spaces and $\mathbb{R}^n$ both have infinite injectivity radius, the theorem is true for spheres of any radius in these spaces. A special case of Theorem 1.1 for symmetric spaces is:

COROLLARY 1.2: *Let X be a non-compact symmetric space of any real rank, and let P_μ be a Pompeiu transform on geodesic spheres in X of radius $r > 0$ with weight function μ that is real analytic and never zero. Let D be a closed geodesic ball and let $f \in \mathcal{D}'(X)$ with $P_\mu f(y) = 0$ for all $y \in X$ with $S(y) \cap D = \emptyset$. Assume for some $y_0 \in X$ that $D(y_0)$ is disjoint from $D \cup$*supp f. *Then* supp $f \subset D$.

This corollary follows from Theorem 1.1 because $\mathcal{A} = \{y \in X \mid d(y, D) > r\}$ is connected in X. Shahshahani and Sitaram [19] prove a related support theorem for the transform with canonical measures on geodesic disks and their conclusion is that f is zero outside a larger region than D. Results such as Theorem 1.1 and Corollary 1.2 can eliminate such shadow regions for spherical integrals.

Fritz John proved global uniqueness for integrals over spheres of radius 1 in $\mathbb{R}^3$ in the standard measure [12, p. 115] under the assumption that supp f is disjoint from $|x| \leq 1$. One can prove our theorem for this special case using a local version of his argument [12, below (6.16) plus the perturbation argument (6.18)] and then our proof (3.1)–(3.2) and the following.

If a distribution, f, has zero averages on all disks of two "well chosen" radii that lie inside a larger disk, B, then the "local injectivity theorem" of [1] states that $f = 0$ in B. Under the extra rather strong assumption that f is supported away from some $D(y_0)$, Theorem 1.1 implies this local theorem in [1] for spheres of only one radius. The next local injectivity theorem (proven in §3) replaces this strong support assumption on f by the requirement that f is supported sufficiently far away from ∂B.

PROPOSITION 1.3: *Let $B \subset \mathbb{R}^n$ be an open ball of radius larger than 3 centered at the origin. Let $f \in \mathcal{D}'(\mathbb{R}^n)$ have support in $\{x \in \mathbb{R}^n \mid |x| < 2\}$. Let P_μ be a Pompeiu transform integrating over spheres of radius $r = 1$ with nowhere zero real analytic weight μ. If $P_\mu f(y) = 0$ when $S(y) \subset B$, then $f \equiv 0$.*

This research is based on the pioneering work of Guillemin [8, 9] that uses microlocal analysis to understand Radon transforms. Sunada [20] and Tsujishita [22] have proven that P_μ is a Fourier integral operator (see also [8]). In our proof of Theorem 1.1, the theory of real analytic Fourier integral operators is used to deduce analytic smoothness of a distribution f from support restrictions on $P_\mu f$ (Proposition 2.1). Then a theorem of Hörmander Kawai, and Kashiwara [11, 13] about analytic singularities and support is used to deduce support restrictions on f from analytic smoothness of f (Lemma 3.1). These proofs follow naturally from the ideas in [5, 6, 15]. The microlocal analysis is given in §2; and the support theorem is proven in §3.

2. The Pompeiu transform as an analytic Fourier integral operator

The analytic wave front set, $\mathrm{WF_A}(f)$, of a distribution $f \in \mathcal{D}'(M)$ is defined using analytic local coordinates and the definition in [11] or [21] for $\mathbb{R}^n$.

PROPOSITION 2.1: *Let M be a real analytic Riemannian manifold and let $\mathcal{A} \subset M$ be open. Let $r > 0$ and assume for each $y \in B(\mathcal{A}, r)$, $I_y > r$. Let P_μ be a Pompeiu* *transform on geodesic spheres in M of radius r with nowhere zero real analytic function μ. Let $f \in \mathcal{D}'(M)$ and let $y_1 \in \mathcal{A}$. Assume $P_\mu f(y) = 0$ for all y in an open neighborhood of y_1. Let $N^*S(y_1)$ be the conormal bundle of $S(y_1)$ in T^*M. Let x and x_a be antipodal points in $S(y_1)$. Let (x,η) and (x_a,η') be covectors in $N^*S(y_1) \setminus 0$. If $(x_a, \pm\eta') \notin \mathrm{WF_A}(f)$, then $(x,\eta) \notin \mathrm{WF_A}(f)$.*

This proposition can be applied if f is zero on an open neighborhood of x_a, because, in this case, there is no wavefront set above x_a.

Proof of Proposition 2.1: The key to the proof of Proposition 2.1 is an understanding of the microlocal properties of the operator P_μ. The relevant microlocal diagram is:

$$\begin{array}{ccc} \Gamma = N^*Z \setminus 0 & \xrightarrow{\rho} & T^*(\mathcal{A}) \setminus 0 \\ \downarrow \pi & & \\ T^*(M) \setminus 0 & & \end{array} \tag{2.1}$$

where π and ρ are the natural projections, where

$$Z = \{(x,y) \in M \times \mathcal{A} \mid d(x,y) = r\}$$

is the **incidence relation** for P_μ, and where N^*Z is the conormal bundle of Z in $T^*(M \times \mathcal{A})$.

It is known [8, 20 top paragraph and remark on p. 488] that P_μ is a Fourier integral operator associated with the Lagrangian manifold Γ. P_μ is analytic elliptic because the weight μ is analytic and nowhere zero (see, e.g., [15]).

To prove the statements about microlocal singularities in Proposition 2.1, one must show that

(2.2) ρ is a two to one local diffeomorphism that maps corresponding covectors in Γ that lie above antipodal points in $S(y_1)$ to the same point in $T^*\mathcal{A}$.

The corresponding covectors are π^{-1} of corresponding covectors (x,η) and (x_a,η') in $N^*S(y_1) \setminus 0$. Under the assumptions of the proposition, the calculus of analytic elliptic Fourier integral operators [18, 21] immediately implies the microlocal smoothness assertion of the theorem because $(x_a, \pm\eta') \notin \mathrm{WF_A}(f)$ and so singularities at x_a do not cancel singularities at x. Therefore, if $P_\mu f = 0$ then $(x,\eta) \notin \mathrm{WF_A}(f)$.

Because the proof of (2.2) is so simple in local coordinates, we will outline it. We use geodesic normal coordinates to write the distance on M locally in terms of the Euclidean distance on a tangent space. Let $y_1 \in \mathcal{A}$ and let B be an open ball centered at $0 \in T_{y_1}M$ of radius $s \in (r, I_{y_1})$ (recall $\forall y \in \mathcal{A},\ \ r < I_y$). Then in geodesic coordinates on B, $\exp = \exp_{y_1} : B \to M$ is a diffeomorphism onto the open ball $\mathcal{B} \subset M$ centered at y_1 of radius s [14, IV 3.4]. Let $U \subset B$ be a neighborhood of zero such that B is a normal neighborhood of each vector in U. This is possible because I_y is lower semicontinuous. Let $\mathcal{U} = \exp_{y_1} U$. By [14, III 8.3 and IV 3.4], the shortest geodesic in M between each point $x \in \mathcal{U}$ and each point $y \in \mathcal{B}$ lies in $\mathcal{B}$, and the proof of [14, IV 3.6] shows that the square of the distance function, $d^2(x,y)$, is smooth on $\mathcal{U} \times \mathcal{B}$. If $X \in U$ and $Y \in B$, let $x = \exp(X)$ and $y = \exp(Y)$. In this case, one can write the distance function on $\mathcal{U} \times \mathcal{B}$ in terms of the Euclidean metric on $T_{y_1}M$ as $d^2(x,y) = ||X - Y||^2 + c(X,Y)$ for some real analytic function c satisfying $\nabla_X c(0,Y) = \nabla_Y c(0,Y) \equiv 0\ \forall Y \in B$ where ∇_X and ∇_Y are the gradients in the respective variables.* These coordinates now give local coordinates on N^*Z in which the needed properties of ρ are easily checked. ∎

3. Proofs

Proof of Theorem 1.1: Let $y_0 \in \mathcal{A}$ as in the statement of Theorem 1.1 and assume the conclusion of the theorem is false. Let $y_2 \in \mathcal{A}$ be such that $D(y_2) \cap \operatorname{supp} f \neq \emptyset$, and let p: $[0,1] \to \mathcal{A}$ be a continuous path from y_0 to y_2. Now choose $\epsilon > 0$ so that:

(3.1) if $y_1 \in M$ and $d(y_1, p([0,1])) \leq \epsilon$, then $y_1 \in \mathcal{A}$ and $r + \epsilon < I_{y_1}$;

(3.2) if $d(x, y_0) \leq r + \epsilon$, then $x \notin \operatorname{supp} f$.

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* By [14, IV 3.4], $\nabla_Y c(0,Y) \equiv 0$ for $Y \in B$. To see $\nabla_X c(0,Y) \equiv 0$ first note that the segment between 0 and $Y \in B \setminus 0$ corresponds to the geodesic between y_1 and $\exp(Y)$. Let S be the inverse image under $\exp_{y_1}$ of the geodesic sphere of radius $||Y||$ centered at $\exp(Y)$; $S = \{X \in B \mid d(\exp(X), \exp(Y)) = ||Y||\}$ and $0 \in S$. By an assumption of Theorem 1.1, $I_{\exp_{y_1}(Y)} > ||Y||$, and so Gauss' Lemma [14, IV 3.3] can be used to conclude that S is perpendicular at $X = 0$ to this segment. (A simple argument using Gauss' lemma and a small geodesic sphere tangent to S at 0 shows that the segment is perpendicular to S even without the assumption about $I_{\exp_{y_1}(Y)}$.) Furthermore, the Euclidean sphere $S' = \{X \in B \mid ||X - Y|| = ||Y||\}$ is also perpendicular at $X = 0$ to this segment. Therefore, S and S' are tangent at $X = 0$ and so directional derivatives of c at $X = 0$ tangent to S are zero. The directional derivative of c at $X = 0$ in the perpendicular direction (in the direction of Y) is zero because geodesics through y_1 correspond to straight lines through the origin in B in the Euclidean distance.

Note that (3.1) can be satisfied because $\mathcal{A}$ is open, $r < I_y \; \forall y \in \mathcal{A}$, and the function I_y is lower semicontinuous (see first footnote). If $t \in [0,1]$ define $T(t)$ to be the closed ball centered at $p(t)$ of radius $r+\epsilon$. By (3.2), $T(0)$ is disjoint from supp f and by assumption, $T(1)$ meets supp f. Let t_1 be the smallest value of $t \in [0,1]$ such that $T(t)$ meets supp f. By the choice of t_1, $T_1 = T(t_1)$ meets supp f only on the boundary, ∂T_1. Let $x \in \partial T_1 \cap \text{supp } f$. Then by (3.1) the point, y_1, that is ϵ units from $p(t_1)$ on the geodesic between $p(t_1)$ and x is in $\mathcal{A}$. The sphere $S(y_1)$ is contained in T_1, and by Gauss' Lemma [14, IV 3.3] and (3.1), $S(y_1)$ is tangent to ∂T_1 at x. Let $\eta \in N_x^* S(y_1) \setminus 0$; then, because $S(y_1)$ and ∂T_1 are tangent at x, $\eta \in N_x^*(\partial T_1)$. But, f is zero near the antipodal point in $S(y_1)$ to x because this antipodal point is in the interior of T_1. Therefore, $S(y_1)$ satisfies the hypotheses of Proposition 2.1 and $(x,\eta) \notin \mathrm{WF_A}(f)$. A special case of a theorem of Hörmander, Kawai, and Kashiwara [11, Theorem 8.5.6] is the final key to the proof:

LEMMA 3.1: *Let $h \in \mathcal{D}'(M)$ and assume h is zero on the interior of T_1. If $x \in \partial T_1 \cap \text{supp } f$ and $(x,\eta) \in N^*(\partial T_1) \setminus 0$, then $(x,\eta) \in \mathrm{WF_A}(f)$.*

By Lemma 3.1, $x \notin$ supp f, but this contradicts the assumption that $T(1)$ meets supp f. ∎

Proof of Proposition 1.3: The proof is done in three steps. Let $S(y)$ be the sphere of radius 1 centered at y. Let $\mathcal{S}_r$ be the sphere centered at the origin of radius r and let $N_r^* = N^*\mathcal{S}_r \setminus 0 \subset T^*(\mathbb{R}^n)$. First, let $r \in [0,1]$. We now show that $N_r^* \cap \mathrm{WF_A}(f) = \emptyset$. Let $(x_0,\xi_0) \in N_r^*$ and choose y so that $|y| = r+1$ and $S(y)$ is tangent to $\mathcal{S}_r$ at x_0. Then, the antipodal point to x_0 in $S(y)$ is not in supp f. Therefore, by Proposition 2.1, $(x_0,\xi_0) \notin \mathrm{WF_A}(f)$. Second, let $r \in (1,2]$. Let $(x_0,\xi_0) \in N_r^*$. Choose y so $|y| = r-1$ and such that $S(y)$ is tangent to $\mathcal{S}_r$ at x_0. Let x_a be the antipodal point to x_0 on $S(y)$. As $(x_a, \pm\xi_0) \notin \mathrm{WF_A}(f)$ by the first step, one can use Proposition 2.1 to show $(x_0,\xi_0) \notin \mathrm{WF_A}(f)$. So, $N_r^* \cap \mathrm{WF_A}(f) = \emptyset$ for all $r \in [0,2]$. Third, use Hörmander's Theorem 8.5.6 [11] (essentially Lemma 3.1 but with $f = 0$ *outside* the sphere) starting at $\mathcal{S}_2$ and working down to $\mathcal{S}_0$ to infer that $f \equiv 0$. ∎

EXAMPLE 3.2: *Let P be the Pompeiu transform that integrates over spheres of radius $r = 1$ in $\mathbb{R}^n$ in standard measure. We construct a smooth function $F(x)$ with $PF \equiv 0$ but supp F is disjoint from the sphere $S(0)$.*

This example shows that the hypotheses of Theorem 1.1 that $D(y_0)$ is disjoint from supp F cannot be weakened to become "$S(y_0)$ is disjoint from supp F." The reason is that the hypothesis about antipodal points in Proposition 2.1 is necessary. We choose an $\epsilon \in (0,1)$ and construct the function F to be zero on $A = \{x \in \mathbb{R}^n \mid 1-\epsilon < |x| < 1+\epsilon\}$, but to have $\partial A \subset$ supp F. Therefore, if $|y| = \epsilon$, then $S(y)$ is tangent to ∂(supp F) at antipodal points x and x_a. Furthermore, covectors in $\mathrm{WF_A}(F)$ at x_a cancel covectors in $\mathrm{WF_A}(F)$ at x to make PF real analytic (in fact, zero) near y, even though F is not analytic in the conormal directions to ∂supp F at either x or at x_a. A related counterexample, [1, Theorem 10], is given to their main theorem for the Pompeiu transform on disks of two radii. Furthermore, John [12, p. 115] constructs a function $f \in C(\mathbb{R}^3)$ for which $PF \equiv 0$ and the interior of $D(0)$ is disjoint from supp F.

Construction: Let k: $[0,\infty) \to \mathbb{R}$ be continuous. Then, for $r \in [0,\infty)$ we define Pompeiu transform $Pk(r)$ to be $P[k(|x|)](y)$ for any y with $|y| = r$. This is well defined because $k(|x|)$ is a radial function and P is rotation invariant.

We define F in intervals. Let $\epsilon \in (0,1)$. First we specify a smooth function f_1 with supp $f_1 \subset [0, 1-\epsilon]$. Next we use f_1 to construct the smooth function f_3 with supp $f_3 \in [1+\epsilon, 3+\epsilon]$ by solving an integral equation. Then we use f_3 to define f_5 with supp $f_5 \subset [3+\epsilon/2, 5+\epsilon/2]$....... Finally we let $F(x) = f_1(|x|) + f_3(|x|) + f_5(|x|) + \cdots$.

Let $g \in C^\infty(\mathbb{R})$ be even, non-zero, and non-negative with

$$\text{supp } g = [-1+\epsilon, 1-\epsilon].$$

Let $f_1(s) = g(s)$ for $s \geq 0$. Now, $f_3(s)$ is constructed. Because f_1 is smooth,

$$Pf_1(r) \text{ is smooth} \quad \text{and} \quad \text{supp } Pf_1 = [\epsilon, 2-\epsilon]. \tag{3.3}$$

If $f(s)$ is a function supported in $[1+\epsilon, 3+\epsilon]$ then a simple exercise using the law of cosines shows for $w = r+1 \in [1+\epsilon, 3+\epsilon]$ that

(3.4)
$$Pf(w-1) = 2^{3-n}(w-1)^{2-n}\Omega_{n-1} \int_{1+\epsilon}^{w} s\left[(w+s)(s^2-(w-2)^2)\right]^{\frac{n-3}{2}}(w-s)^{\frac{n-3}{2}} f(s)ds$$

where Ω_{n-1} is the area of the unit sphere in $\mathbb{R}^{n-1}$. Because of (3.4), the equation

$$-Pf_1(w-1) = [P\overline{f}_3](w-1) \tag{3.5}$$

is a generalized first kind Volterra or Abel integral equation for $\overline{f}_3(s)$ depending on whether $\frac{n-3}{2}$ is an integer or half-integer. By (3.3), the function $Pf_1(w-1)$ is zero to infinite order at $w = 1+\epsilon$. Therefore, the integral equation (3.5) satisfies the hypotheses of theorems A or B [16] or the reduction on pp. 515–16 [16] (see also [23]). So, one can solve (3.5) for the function $\overline{f}_3(s)$ for $s \in [1+\epsilon, 3+\epsilon)$. As Pf_1 is smooth, $\overline{f}_3$ is smooth on $[1+\epsilon, 3+\epsilon)$ [23]. As $r = \epsilon \in$ supp Pf_1, $s = 1+\epsilon \in$ supp $\overline{f}_3$. Since $Pf_1(w-1)$ is zero to infinite order at $w = 1+\epsilon$, $\overline{f}_3(s)$ is zero to infinite order at $s = 1+\epsilon$; so, $\overline{f}_3$ can be smoothly extended to $[0, 3+\epsilon)$ to be zero on $[0, 1+\epsilon]$. We define f_3 to be $\overline{f}_3$ on $[0, 3+\epsilon/2]$ and to taper smoothly off to zero on $[3+\epsilon/2, 3+\epsilon]$. Therefore, f_3 is smooth, supp $f_3 \subset [1+\epsilon, 3+\epsilon]$, and by (3.5),

$$P(f_1+f_3)(w-1) = 0 \quad \text{for } w \in [0, 3+\epsilon/2]. \tag{3.6}$$

We construct f_5 by first solving

$$-Pf_3(w-1) = P\overline{f}_5(w-1) \tag{3.7}$$

for $\overline{f}_5$. This is an integral equation for $\overline{f}_5$ like (3.4-5) but with lower limit $3+\epsilon/2$. Using the analogous argument to the above, one can solve (3.7) for $\overline{f}_5(s)$ for $s \in [3+\epsilon/2, 5+\epsilon/2)$. Because $Pf_3(w-1)$ is zero to infinite order at $w = 3+\epsilon/2$, $\overline{f}_5(s)$ can be extended smoothly to $[0, 5+\epsilon/2)$ to be zero on $[0, 3+\epsilon/2]$. Now define f_5 by tapering off $\overline{f}_5$ to zero in $[5+\epsilon/4, 5+\epsilon/2]$. As above, f_5 is smooth, supp $f_5 \subset [3+\epsilon/2, 5+\epsilon/2]$, and $P(f_1+f_3+f_5)(w-1) = 0$ for $w \in [0, 5+\epsilon/4]$. This argument can be repeated to finish the construction of F. (To construct f_7 one starts by solving $-P(f_3+f_5) = P\overline{f}_7$ on $[5+\epsilon/4, 7+\epsilon/4)$.) ∎

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